Mathematics: The Core Course for A-level; L. Bostock and S. Chandler

Chapter 9: Integration

Miscellaneous Exercises 9

1 a) $I = \int_1^3 \frac{x+2}{x+1}\,dx$.

By long division we have $\frac{x+2}{x+1} = 1 + \frac{1}{x+1}$

$$\therefore I = \int_1^3 1 + \frac{1}{x+1}\,dx = \Big[x + \ln|x+1|\Big]_1^3$$

$$= \big[3 + \ln(4)\big] - \big[1 + \ln(2)\big]$$

$$= 2 + \ln\frac{4}{2} = 2 + \ln 2$$

b) $I = \int_0^{\pi/2} \sin^4 x \cdot \cos x\,dx$

Let $u = \sin x$; Then $du = \cos x\,dx$

Also $u = \sin 0 = 0$ and $u = \sin\frac{\pi}{2} = 1$

So $I = \int_0^1 u^4\,du = \frac{1}{5}u^5\Big|_0^1 = \frac{1}{5}$.

② ⓐ $I = \int_0^1 \frac{x^2}{(1+x^3)^2}\,dx$

Let $u = 1+x^3$; Hence $du = 3x^2\,dx$

$\therefore \quad \frac{1}{3}\,du = x^2\,dx$

$\therefore \quad I = \int_a^b \frac{1}{3}\cdot\frac{1}{u^2}\,du$ where $\left.\begin{array}{l} a = 1+0^3 = 1 \\ b = 1+1^3 = 2 \end{array}\right\}$ FROM $u = 1+x^3$

So $I = \int_1^2 \frac{1}{3u^2}\,du = \left[-\frac{1}{3u}\right]_1^2 = \left(-\frac{1}{6}\right)-\left(-\frac{1}{3}\right)$

$= \frac{1}{6} \simeq 0.167$ (3 d.p.)

ⓑ $I = \int_5^6 \frac{1}{(x-2)(x-4)}\,dx$

By partial fractions : $\frac{1}{(x-2)(x-4)} = \frac{A}{(x-2)} + \frac{B}{(x-4)}$

So $\quad 1 = A(x-4) + B(x-2)$

$x=2: \quad 1 = -2A \Rightarrow A = -\frac{1}{2}$

$x=4: \quad 1 = 2B \Rightarrow B = \frac{1}{2}$

So $I = \int_5^6 \frac{1}{(x-2)(x-4)}\,dx = \int_5^6 \frac{-\frac{1}{2}}{(x-2)} + \frac{\frac{1}{2}}{(x-4)}\,dx$

$= \left[-\frac{1}{2}\ln(x-2) + \frac{1}{2}\ln(x-4)\right]_5^6$

$= \frac{1}{2}\ln\left(\frac{x-4}{x-2}\right)\Big]_5^6 = \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{3}$.

$= \frac{1}{2}\ln\left(\frac{3}{2}\right) \doteqdot 0.203$

③ ⓐ $I = \int x(1+x^2)^5 \, dx$

let $u = 1+x^2$; $\therefore \; du = 2x \, dx$

$\Rightarrow \frac{1}{2} du = x \, dx$

So $I = \int \frac{1}{2} \cdot u^5 \, du = \frac{1}{12} u^6 + c$

$= \frac{1}{12} (1+x^2)^6 + c$

ⓑ $I = \int x \cdot e^{-2x} \, dx$

let $u = x$, $\therefore \; du = 1 . dx$

$dv = e^{-2x}$, $\therefore \; v = -\frac{1}{2} e^{-2x}$

So $I = -\frac{1}{2} x . e^{-2x} + \frac{1}{2} \int e^{-2x} \, dx$

$= -\frac{1}{2} x . e^{-2x} - \frac{1}{4} e^{-2x} + c$

ⓒ $I = \int_2^3 \frac{1}{x(x^2-1)} \, dx$

By Partial fractions : $\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)}$

$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

So $\quad 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

when $x = 0: \quad 1 = -A \Rightarrow A = -1$

$x = 1: \quad 1 = 2B \Rightarrow B = \frac{1}{2}$

$x = -1: \quad 1 = 2C \Rightarrow C = \frac{1}{2}$

So
$$I = \int_2^3 -\frac{1}{x} + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\, dx$$

$$= \left[-\ln x + \tfrac{1}{2}\ln|x-1| + \tfrac{1}{2}\ln|x+1|\right]_2^3$$

$$= \left[-\ln 3 + \tfrac{1}{2}\ln(3^2-1)\right] - \left[-\ln 2 + \tfrac{1}{2}\ln(2^2-1)\right]$$

$$= \tfrac{1}{2}\ln\frac{32}{27}$$ (Note that $\ln\frac{2}{3} = \frac{1}{2}\ln\frac{4}{9}$)

(d)
$$I = \int_0^{\pi/2} \sin^2 x \cdot \cos^3 x\, dx$$

$$= \int_0^{\pi/2} \sin^2 x \cdot (1 - \sin^2 x)\cos x\, dx$$

$$= \int_0^{\pi/2} \sin^2 x \cdot \cos x - \sin^4 x \cdot \cos x\, dx$$

let $u = \sin x$; $\therefore du = \cos x\, dx$

Also $u_1 = \sin 0 = 0$ and $u_2 = \sin\frac{\pi}{2} = 1$

So
$$I = \int_0^1 u^2 - u^4\, du = \left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right]_0^1 = \frac{1}{3} - \frac{1}{5}$$
$$= \frac{2}{15}.$$

(4) (a) (i) Find $I = \int (e^{2x}-1)^2 \, dx$

$$= \int e^{4x} - 2e^{2x} + 1 \, dx$$

$$= \frac{1}{4} e^{4x} - e^{2x} + x + c$$

(ii) $I = \int \frac{\cos\theta}{\sqrt{\sin\theta}} \, d\theta$

let $u = \sin\theta$, $\therefore$ $du = \cos\theta \, d\theta$

So $I = \int \frac{1}{\sqrt{u}} \, du = \int \frac{1}{u^{1/2}} \, du = -2u^{-1/2} + c$

$$= -2(\sin\theta)^{-1/2} + c$$

$$= -\frac{2}{\sqrt{\sin\theta}} + c$$

(b) (i) $I = \int_0^{\pi/2} 2\cos 2\theta \cos\theta \, d\theta$

$\cos A + \cos B = 2\cos\frac{A+B}{2} \cdot \cos\frac{A-B}{2}$

So $\frac{A+B}{2} = 2\theta$ & $\frac{A-B}{2} = \theta$ $\Rightarrow A = 3\theta$, $B = \theta$

$\therefore$ $I = \int_0^{\pi/2} \cos 3\theta + \cos\theta \, d\theta = \left[\frac{1}{3}\sin 3\theta + \sin\theta\right]_0^{\pi/2}$

$$= \left(-\frac{1}{3} + 1\right) - (0 + 0)$$

$$= \frac{2}{3}$$

(ii) $I = \int_3^4 \frac{5}{x^2+x-6}\,dx = \int_3^4 \frac{5}{(x+3)(x-2)}\,dx$

Partial fractions : $\frac{5}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$

So $5 = A(x-2) + B(x+3)$

When $x = 2$: $5 = 5B \Rightarrow B = 1$
$x = -3$: $5 = -5A \Rightarrow A = -1$

So $I = \int_3^4 -\frac{1}{x+3} + \frac{1}{x-2}\,dx$

$= \left[-\ln|x+3| + \ln|x-2|\right]_3^4 = \left[\ln\left|\frac{x-2}{x+3}\right|\right]_3^4$

$= \ln\frac{2}{7} - \ln\frac{1}{6} = \ln\frac{12}{7}$

(5) (a) $I = \int_0^{\pi/6} \tan 2\theta\,d\theta = \int_0^{\pi/6} \frac{\sin 2\theta}{\cos 2\theta}\,d\theta$

Let $u = \cos 2\theta$, $\therefore$ $du = -2\sin 2\theta\,d\theta$

$\Rightarrow -\frac{1}{2}\,du = \sin 2\theta\,d\theta$

Also $u = 1$ when $\theta = 0$ & $u = \frac{1}{2}$ when $\theta = \frac{\pi}{6}$

So $I = \int_1^{\frac{1}{2}} -\frac{1}{2} \frac{1}{u} \, du = \left[-\frac{1}{2} \ln u\right]_1^{\frac{1}{2}}$

$= -\frac{1}{2} \ln \frac{1}{2} - \left(-\frac{1}{2} \ln 1\right)$

$= -\frac{1}{2} \ln \frac{1}{2}$

$= -\frac{1}{2} \left[\ln 1 - \ln 2\right]$

$= \frac{1}{2} \ln 2.$

(b) $I = \int_0^1 x(1-x)^{\frac{1}{2}} \, dx$

let $u = 1-x$, $\therefore du = -dx$

$\Rightarrow -du = dx$

Also when $x=0$, $u=1$ & when $x=1$, $u=0$

So $I = -\int_1^0 (1-u)\, u^{\frac{1}{2}} \, du = \int_0^1 (1-u) u^{\frac{1}{2}} \, du$

$= \int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}} \, du$

$= \left[\frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}}\right]_0^1$

$= \frac{4}{15}$

(6) (a) $I = \int_0^2 x(x^2+1)^3 \, dx$

let $u = x^2+1$, $\therefore du = 2x \, dx$

$\Rightarrow \frac{1}{2} du = x \, dx$

& $x=0 \Rightarrow u=1$, $x=2 \Rightarrow u=5$

So $I = \int_0^5 \frac{1}{2} (u)^3 \, du = \left[\frac{1}{8} u^4 \right]_0^5 = \frac{625}{8}$

(b) $I = \int_2^3 \frac{1}{(4-x)(x-1)} \, dx$

Partial fractions : $\frac{1}{(4-x)(x-1)} = \frac{A}{4-x} + \frac{B}{x-1}$

$1 = A(x-1) + B(4-x)$

So $x=1$: $1 = 3B \Rightarrow B = 1/3$

$x=4$: $1 = 3A \Rightarrow A = 1/3$

So $I = \int_2^3 \frac{1/3}{4-x} + \frac{1/3}{x-1} \, dx = \left[-\frac{1}{3} \ln|4-x| + \frac{1}{3} \ln|x-1| \right]_2^3$

$= \left[\frac{1}{3} \ln \left| \frac{x-1}{4-x} \right| \right]_2^3$

$= \frac{1}{3} \ln\left(\frac{2}{1}\right) - \frac{1}{3} \ln\left(\frac{1}{2}\right) = \frac{1}{3} \ln 4.$

(c) $I = \int_0^{\pi/2} \tan \frac{1}{2}x \, dx = \int_0^{\pi/2} \frac{\sin \frac{1}{2}x}{\cos \frac{1}{2}x} \, dx$

Let $u = \cos \frac{1}{2} x$, $\therefore du = -\frac{1}{2} \sin \frac{1}{2} x \, dx$

$\Rightarrow -2du = \sin \frac{1}{2} x \, dx$

So $\int -\frac{2}{u} \, du = -2 \ln |u|$

$\Rightarrow I = \int_0^{\pi/2} \tan \frac{1}{2} x \, dx = \left[-2 \ln \left| \cos \frac{1}{2} x \right| \right]_0^{\pi/2}$

$= -2 \ln \left(\frac{\sqrt{2}}{2} \right) - (-2 \ln 1)$

$= -\ln \left(\frac{2}{4} \right) = -\ln \frac{1}{2}$

$= \ln 2$

(7) (a) $I = \int_{\pi/6}^{\pi/4} 2 \sin 3x \cos 2x \, dx$

By trig identities $2 \sin 3x \cos 2x = 2 \sin \frac{A+B}{2} \cdot \cos \frac{A-B}{2}$

So $\frac{A+B}{2} = 3x$ & $\frac{A-B}{2} = 2x$

$\Rightarrow A = 5x$ & $B = x$

So $I = \int_{\pi/6}^{\pi/4} \sin 5x + \sin x \, dx$

$$I = \left[-\frac{1}{5}\cos 5x - \cos x \right]_{\pi/6}^{\pi/4}$$

$$\simeq -0.56568 + 0.69282$$

$$\simeq 0.12714$$

(b) $$I = \int \frac{1}{4\cos^2 x - 9\sin^2 x}\, dx$$

If $t = \tan x$, $\quad 1 + t^2 = \tan^2 x + 1 = \sec^2 x$

$$= \frac{1}{\cos^2 x}$$

$$\therefore \quad \cos^2 x = \frac{1}{1+t^2}$$

& By $\cos^2 x + \sin^2 x = 1$ we have

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$$

Also $\quad t = \tan x \Rightarrow dt = \sec^2 x \, dx$

$$= (1+t^2)\, dx$$

$$\text{So} \quad \frac{1}{1+t^2}\, dt = dx$$

So $$I = \int \frac{1}{1+t^2} \cdot \frac{1}{\frac{4}{1+t^2} - \frac{9t^2}{1+t^2}}\, dt$$

$$I = \int \frac{1}{4-9t^2}\,dt$$

$$= \int \frac{1}{(2-3t)(2+3t)}\,dt$$

Partial fractions : $\frac{1}{(2-3t)(2+3t)} = \frac{A}{2-3t} + \frac{B}{2+3t}$

$$\therefore \quad 1 = A(2+3t) + B(2-3t)$$

if $t = \frac{2}{3}$: $\quad 1 = 4A \Rightarrow A = 1/4$

$t = -\frac{2}{3}$: $\quad 1 = 4B \Rightarrow B = 1/4$

So

$$I = \int \frac{1/4}{2-3t} + \frac{1/4}{2+3t}\,dt$$

$$= -\frac{1}{4}\cdot\frac{1}{3}\ln|2-3t| + \frac{1}{4}\cdot\frac{1}{3}\ln|2+3t|$$

$$= \frac{1}{12}\ln\left|\frac{2+3t}{2-3t}\right|$$

$$= \frac{1}{12}\ln\left|\frac{2+3\tan x}{2-3\tan x}\right| + c$$

(8) (a) $I = \int_1^2 \frac{(x^2-1)^2}{x^3}\,dx$

$= \int_1^2 \frac{x^4 - 2x^2 + 1}{x^3}\,dx$

$= \int_1^2 x - \frac{2}{x} + \frac{1}{x^3}\,dx$

$= \left[\frac{x^2}{2} - 2\ln x - \frac{1}{2}\frac{1}{x^2} \right]_1^2$

$= \left(2 - 2\ln 2 - \frac{1}{2}\cdot\frac{1}{4}\right) - \left(\frac{1}{2} - 2\ln 1 - \frac{1}{2}\right)$

$= \frac{15}{8} - 2\ln 2 = \frac{15}{8} - \ln 4$

(b) $I = \int_{\frac{\pi}{2}}^{\pi} \sin 3x + \cos \frac{1}{2}x\,dx$

$= \left[-\frac{1}{3}\cos 3x + 2\sin \tfrac{1}{2}x \right]_{\pi/2}^{\pi}$

$= \left(\frac{1}{3} + 1\right) - \left(0 + 2\frac{\sqrt{2}}{2}\right)$

$= 1\tfrac{1}{3} - \sqrt{2}.$

(c) $I = \int_1^2 \frac{x+2}{x(x+4)} \, dx$

Partial fractions: $\frac{x+2}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4}$

$$x+2 = A(x+4) + Bx$$

let $x = 0$: $2 = 4A \Rightarrow A = \frac{1}{2}$

$x = -4$: $-2 = -4B \Rightarrow B = \frac{1}{2}$

So $I = \int_1^2 \frac{1/2}{x} + \frac{1/2}{x+4} \, dx$

$$= \left[\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x+4|\right]_1^2$$

$$= \left(\frac{1}{2}\ln 2 + \frac{1}{2}\ln 6\right) - \left(\frac{1}{2}\ln 1 + \frac{1}{2}\ln 5\right)$$

$$= \frac{1}{2}\ln\frac{12}{5}$$

(9) (a) $I = \int \frac{x}{x^2-2x-3} \, dx$

Partial fractions: $\frac{x}{x^2-2x-3} = \frac{x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

So $x = A(x+1) + B(x-3)$

when $x = 3$: $3 = 4A \Rightarrow A = 3/4$

$x = -1$: $-1 = -4B \Rightarrow B = 1/4$

$$\therefore \quad I = \int \frac{3/4}{x-3} + \frac{1/4}{x+1}\, dx$$

$$= \frac{3}{4}\ln|x-3| + \frac{1}{4}\ln|x+1| + c$$

$$= \frac{1}{4}\ln\left|(x-3)^3(x+1)\right| + c$$

(b) $I = \int_0^1 x^2 (1-x)^{\frac{1}{2}}\, dx$

let $z = 1-x$, $\therefore dz = -dx \Rightarrow -dz = dx$

and when $x=0$, $z=1$;
$x=1$ $z=0$;

$$\therefore \quad I = -\int_1^0 (1-z)^2 \cdot z\, dz = \int_0^1 z(1-z)^2\, dz$$

$$= \int_0^1 z - 2z^2 + z^3\, dz$$

$$= \left[\frac{z^2}{2} - \frac{2}{3}z^3 + \frac{1}{4}z^4\right]_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}.$$

(c) $\frac{dy}{dx} = 1-y$ when $y<1$, & $y=0$ when $x=0$

So $\int \frac{1}{1-y}\, dy = \int dx \Rightarrow \ln|1-y| = x + c$

At $(0,0)$ $\therefore$ $\ln 1 = 0 + c \Rightarrow c = 0$, $\therefore \ln|1-y| = x$

$$\Rightarrow y = 1 - e^x.$$

(10) (a) $\int_0^1 \frac{2x^2}{2x+1} dx$

Do long division:

$$\begin{array}{r} x \quad - 1/2 \\ 2x+1 \overline{)\, 2x^2 \qquad\quad} \\ \underline{2x^2 + x \quad} \\ -x \qquad \\ \underline{-x - \tfrac{1}{2}} \\ \tfrac{1}{2} \end{array}$$

So $\int_0^1 \frac{2x^2}{2x+1} dx = \int_0^1 (x-\tfrac{1}{2}) + \frac{\tfrac{1}{2}}{2x+1} dx$

$= \left[\frac{x^2}{2} - \frac{1}{2}x + \frac{1}{2} \cdot \frac{1}{2} \ln|2x+1| \right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4} \ln 3 \right) - 0 \simeq 0.275$

(b) $I = \int_0^{\pi/3} \sin^2 x \cdot \cos^2 x \, dx = \int_0^{\pi/3} (\sin x \cdot \cos x)^2 dx$

$= \int_0^{\pi/3} (\tfrac{1}{2} \sin 2x)^2 dx$

$= \frac{1}{4} \int_0^{\pi/3} \sin^2 2x \, dx$

By $\cos 2x = 1 - 2\sin^2 x$ we have $\sin^2 2x = \frac{1}{2} \cdot (1 - \cos 4x)$, so

$I = \frac{1}{4} \int_0^{\pi/3} \frac{1-\cos 4x}{8} dx = \frac{1}{8} \left[x - \frac{1}{4} \sin 4x \right]_0^{\pi/3}$

$= \frac{1}{8} \left(\frac{\pi}{3} - \frac{1}{4} \sin \frac{4\pi}{3} \right) = 0.158$. to 3 d.p.

(c) $I = \int_{e}^{e^2} \frac{1}{x.\ln x} dx$

let $u = \ln x$, $\therefore du = \frac{1}{x} dx$

also when $x = e$, $u = 1$
$x = e^2$, $u = 2$

$\therefore I = \int_{1}^{2} \frac{1}{u} du = \ln u \Big|_{1}^{2} = \ln 2 - \ln 1 \simeq 0.693$

(11) $I = \int_{4}^{5} \frac{2x}{x^2 - 4x + 3} dx$

Partial fractions: $\frac{2x}{x^2-4x+3} = \frac{2x}{(x-1)(x-3)}$

$= \frac{A}{x-1} + \frac{B}{x-3}$

So $2x = A(x-3) + B(x-1)$

when $x = 3$: $6 = 2B \Rightarrow B = 3$
$x = 1$: $2 = -2A \Rightarrow A = -1$

So $I = \int_{4}^{5} -\frac{1}{x-1} + \frac{3}{x-3} dx$

$= \Big[-\ln(x-1) + 3\ln(x-3)\Big]_4^5 = \Big[3\ln\left(\frac{x-3}{x-1}\right)\Big]_4^5$

$= 3\ln\frac{1}{2} - 3\ln\frac{1}{3} = 3\ln\frac{3}{2}$

(b) $$I = \int_2^5 \frac{1}{x^2\sqrt{x-1}}\,dx$$

let $x = \sec^2 y$, $\therefore$ $dx = 2\sec y \cdot \tan y \sec y \, dy$

$$= 2\sec^2 y \cdot \tan y \, dy$$

$\therefore$ Also when $x = 2$, $y = \cos^{-1}\frac{1}{\sqrt{2}} = \pi/4$

$x = 5$, $y = \cos^{-1}\frac{1}{\sqrt{5}} = 1.107$

So $$I = \int_{\pi/4}^{1.107} \frac{1}{\sec^4 y\sqrt{\sec^2 y - 1}} \cdot 2\sec^2 y \tan y \, dy$$

$$= 2\int_{\pi/4}^{1.107} \frac{1}{\sec^2 y} \cdot \frac{\tan y}{\sqrt{\tan^2 y}}\, dy$$

$$= 2\int_{\pi/4}^{1.107} \cos^2 y \, dy$$

By $\cos 2y = 2\cos^2 y - 1$ we have $\cos^2 y = \frac{\cos 2y - 1}{2}$

So $$I = 2\int_{\pi/4}^{1.107} \frac{\cos 2y + 1}{2}\, dy$$

$$= \left[\frac{1}{2}\sin 2y + y\right]_{\pi/4}^{1.107} = \left(\frac{1}{2}(0.8) + 1.107\right) - \left(\frac{1}{2}(1) + \frac{\pi}{4}\right)$$

$$= -\frac{\pi}{4} - 0.1 + \cos^{-1}\frac{1}{\sqrt{5}}$$

where $\cos^{-1}\frac{1}{\sqrt{5}} \simeq 1.107$.

(12) (a) $I = \int \frac{2x-1}{(x+1)^2}\,dx$

let $u = x+1$, $\therefore\ du = dx$

also $u-1 = x \Rightarrow 2(u-1)-1 = 2x-1$

So $I = \int \frac{2u-3}{u^2}\,du = \int \frac{2}{u} - \frac{3}{u^2}\,du$

$= 2\ln|u| + \frac{3}{u} + c$

$= 2\ln|x+1| + \frac{3}{x+1} + c$

$I = \int \left(e^x - \frac{1}{e^x}\right)^2 dx$

$= \int (e^x - e^{-x})^2\,dx = \int e^{2x} - 2 + e^{-2x}\,dx$

$= \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x} - 2x$

$= \frac{1}{2}(e^{2x} - e^{-2x} - 4x)$.

(b) $I = \int_0^{\pi/4} \tan^2 x\,dx$

By $1+\tan^2 x = \sec^2 x$ we have $\tan^2 x = \sec^2 x - 1$

$\therefore\ I = \int_0^{\pi/4} \sec^2 x - 1\,dx = \left[\tan x - x\right]_0^{\pi/4} = 1 - \pi/4$

(13)
(a) $I = \int_0^1 \frac{8}{3+4x}\,dx$

let $u = 3+4x$, $\therefore du = 4\,dx$
$\Rightarrow \frac{1}{4}\,du = dx$

And if $x=0$, $u=3$
$x=1$, $u=7$

So $I = \int_3^7 \frac{1}{4}\cdot\frac{8}{u}\,du = \Big[2\ln|u|\Big]_3^7$
$= 2\ln 7 - 2\ln 3$
$= 2\ln\frac{7}{3}$

(b) $I = \int_0^1 \frac{8}{\sqrt{3+4x}}\,dx$

let $u = 3+4x$, $\therefore du = 4\,dx$
$\Rightarrow \frac{1}{4}\,du = dx$

And if $x=0$, $u=3$
$x=1$, $u=7$

So $I = \int_3^7 \frac{1}{4}\cdot\frac{8}{\sqrt{u}}\,du = \Big[2\times 2u^{\frac{1}{2}}\Big]_3^7$
$= 4\sqrt{7} - 4\sqrt{3}$

(c) $I = \int_0^1 \frac{8x}{3+4x}\,dx$

By long division we have

$$\begin{array}{r} 2 \\ 3+4x\,\overline{)\,8x} \\ \underline{8x+6} \\ -6 \end{array}$$

So $I = \int_0^1 2 - \frac{6}{3+4x}\,dx$

$= \left[2x - 6\cdot\frac{1}{4}\ln|3+4x|\right]_0^1$

$= \left(2 - \frac{3}{2}\ln 7\right) - \left(-\frac{3}{2}\ln 3\right)$

$= 2 + \frac{3}{2}\ln\frac{3}{7} \quad = 2 - \frac{3}{2}\ln\frac{7}{3}$

(14) (a) $\tan 2y = e^x \quad \Rightarrow \quad 1 + \tan^2 2y = \sec^2 2y = 1 + e^{2x}$ (*)

$2\sec^2 2y \cdot \frac{dy}{dx} = e^x$ (**)

$2\sec^2 2y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 4\sec 2y \cdot (2\tan 2y \cdot \sec 2y) = e^x$

$\therefore\ 2\sec^2 2y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 8\sec^2 2y \tan 2y = e^x$

By (*) : $2(1+e^{2x})\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 8(1+e^{2x})\cdot e^x = e^x$

By (**) : $2(1+e^{2x})\frac{d^2y}{dx^2} + \frac{e^x}{2(1+e^{2x})}\cdot 8(1+e^{2x})\cdot e^x = e^x$

$$\therefore \quad 2(1+e^{2x})\frac{d^2y}{dx^2} + 4e^{2x} = e^{x}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{e^{x} - 4e^{2x}}{2(1+e^{2x})}.$$

(b) i) $I = \int \frac{1}{\sqrt{x} - x}\, dx$

let $u^2 = x$, $\therefore 2u\,du = dx$

$$\therefore \quad I = \int \frac{1}{u-u^2} \cdot 2u\,du = \int \frac{2}{1-u}\, du$$

$$= -2\ln|1-u| + c$$

$$= -2\ln|1-\sqrt{x}| + c$$

$$= -\ln|1-\sqrt{x}|^2 + c$$

ii) $I = \int (1-3\cos^2 x)^{\frac{1}{2}} \cdot \sin 2x\, dx$

let $u = 1-3\cos^2 x$, $\therefore du = +3 \times 2\cos x \sin x\, dx$

$$= 3 \cdot \sin 2x\, dx$$

$$\Rightarrow \frac{1}{3}du = \sin 2x\, dx$$

$$\therefore \quad I = \int u^{1/2} \cdot \frac{1}{3}\, du$$

$$= \frac{2}{3} u^{3/2} \cdot \frac{1}{3} + c$$

$$= \frac{2}{9}\left(1 - 3\cos^2 x\right)^{3/2} + c$$

15 a $$I = \int_1^2 \frac{(x-1)^2}{x^3}\, dx$$

$$= \int_1^2 \frac{x^2 - 2x + 1}{x^3}\, dx$$

$$= \int_1^2 \frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3}\, dx = \left[\ln|x| + \frac{2}{x} - \frac{1}{2}\cdot\frac{1}{x^2}\right]_1^2$$

$$= \left(\ln 2 + 1 - \frac{1}{8}\right) - \left(0 + 2 - \frac{1}{2}\right)$$

$$\simeq 0.068$$

b $$I = \int_{\alpha}^{7\alpha} \sin 2(x + \alpha)\, dx \quad \text{when } \alpha = \frac{\pi}{24}$$

$$I = \left[-\frac{1}{2}\cos 2\left(x + \frac{\pi}{24}\right)\right]_{\pi/24}^{7\pi/24}$$

$$= -\frac{1}{2}\cos\frac{2\pi}{3} + \frac{1}{2}\cos\frac{\pi}{6} = \frac{1}{2}\left(\frac{1}{2} + \sqrt{3}\right)$$

(c) $I = \int_0^3 (x+1)^{3/2}\, dx$

let $u = x+1$, $\therefore$ $du = dx$

and $x = 0 \Rightarrow u = 1$
$x = 3 \Rightarrow u = 4$

So $I = \int_1^4 u^{3/2}\, du = \frac{2}{5} u^{5/2} \Big|_1^4 = \frac{2}{5}(32) - \frac{2}{5}$

$= 12.4$

Because of the "hence" I won't integrate $\int_0^3 x\sqrt{x+1}\, dx$ directly.

Instead we can do

$$\int_0^3 (x+1)^{3/2}\, dx = \int_0^3 (x+1)(x+1)^{1/2}\, dx$$

$$= \int_0^3 x(x+1)^{1/2}\, dx + \int_0^3 (x+1)^{1/2}\, dx$$

$\therefore$

$$12.4 - \int_0^3 (x+1)^{1/2}\, dx = \int_0^3 x(x+1)^{1/2}\, dx$$

$$12.4 - \left[\frac{2}{3}(x+1)^{3/2}\right]_0^3 = \int_0^3 x(x+1)^{1/2}\, dx$$

$$12.4 - \left(\frac{2}{3} \times 8 - \frac{2}{3}\right) = \int_0^3 x(x+1)^{1/2}\, dx$$

$$12.4 - 4\tfrac{2}{3} = \frac{116}{15} = \int_0^3 x(x+1)^{1/2}\, dx$$

(16) (a) $I = \int 2\cos 3x \cdot \sin x \, dx$

Now, $2\cos 3x \cdot \sin x = 2\cos\left(\frac{A+B}{2}\right)x \cdot \sin\left(\frac{A-B}{2}\right)x = \sin Ax - \sin Bx.$

So $3x = \frac{A+B}{2} \cdot x$ & $x = \frac{A-B}{2} \cdot x$

$\Rightarrow A = 4$ & $B = 2$

So $I = \int \sin 4x - \sin 2x \, dx$

$= -\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x + c$

$I = \int \frac{x-2}{\sqrt{x-1}} \, dx$

let $u = x - 1$, $\therefore du = dx$

& $u - 1 = x - 2$

So $I = \int \frac{u-1}{\sqrt{u}} \, du = \int u^{1/2} - u^{-1/2} \, du$

$= \frac{2}{3}u^{3/2} - 2u^{1/2} + c$

$= \frac{2}{3}(x-1)^{3/2} - 2(x-1)^{1/2} + c$

(b) $I = \displaystyle\int_0^{\pi/3} \frac{1}{9-8\sin^2 x}\,dx$

let $t = \tan x$, $\therefore$ $dt = \sec^2 x\,dx$

$= 1+\tan^2 x \; dx$

So $\dfrac{1}{1+t^2}\,dt = dx$

If $t = \tan x$ Then $\sin x = \dfrac{t}{\sqrt{1+t^2}}$, $\therefore$ $\sin^2 x = \dfrac{t^2}{1+t^2}$

Also, when $x = 0$, $t = 0$

$x = \pi/3$, $t = \sqrt{3}$

So

$$I = \int_0^{\sqrt{3}} \frac{1}{9-8\frac{t^2}{1+t^2}} \cdot \frac{1}{1+t^2}\,dt$$

$$= \int_0^{\sqrt{3}} \frac{1+t^2}{9(1+t^2)-8t^2} \cdot \frac{1}{1+t^2}\,dt$$

$$= \int_0^{\sqrt{3}} \frac{1}{9+t^2}\,dt = \left[\frac{1}{3}\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}} = \frac{1}{3}\cdot\frac{\pi}{6} = \frac{\pi}{18}$$

(Note : The integrand gives a Standard Result. or we could do The Substitution $t = 3\tan\theta$ to get The answer)

(17) (a) $I = \int \frac{x+1}{x(2x+1)}\,dx$

Partial fractions: $\frac{x+1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$

$\therefore \quad x+1 = A(2x+1) + Bx$

when $x=0$: $\quad 1 = A$

$x=-\frac{1}{2}$: $\quad \frac{1}{2} = -\frac{1}{2}B \Rightarrow B = -1$

So $I = \int \frac{1}{x} - \frac{1}{2x+1}\,dx$

$= \ln|x| - \frac{1}{2}\ln|2x+1| + c = \ln\left|\frac{x}{\sqrt{2x+1}}\right| + c$

(b) $I = \int \frac{x(2x+1)}{x+1}\,dx = \int \frac{2x^2+x}{x+1}\,dx$

Long division:

$$\begin{array}{r|l} & 2x - 1 \\ \hline x+1 & 2x^2 + x \\ & 2x^2 + 2x \\ \hline & \quad -x \\ & \quad -x - 1 \\ \hline & \qquad 1 \end{array}$$

So $I = \int 2x - 1 + \frac{1}{x+1}\,dx$

$= x^2 - x + \ln|x+1| + c$

(18) (a) $I = \int_1^a \frac{x^4 - 1}{x^3}\,dx = \frac{9}{8}$

$= \int_1^a x - \frac{1}{x^3}\,dx = \left[\frac{x^2}{2} + \frac{1}{x^2}\cdot\frac{1}{2}\right]_1^a = \frac{9}{8}$

$= \left(\frac{a^2}{2} + \frac{1}{a^2}\cdot\frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{4}\right) = \frac{9}{8}$

$= \frac{a^2}{2} + \frac{1}{2a^2} = \frac{15}{8}$

$= \quad a^4 + 1 = \frac{15}{4}a^2$

$= 4a^4 - 15a^2 + 4 = 0$

$= (4a^2+1)(a^2-4) = 0$

So $4a^2 + 1 = 0 \Rightarrow a^2 = -\frac{1}{4}$: Not valid

or $a^2 - 4 = 0 \Rightarrow a = \pm 2$. But $a > 1$, so $a = 2$.

(b) $I = \int_{\pi/2}^{\pi} \cos nx\,dx = \frac{1}{n}\sin nx\Big|_{\pi/2}^{\pi}$

For n even, $n = 2k$, $k = 0, 1, 2, \ldots$

n odd $n = 2k+1$, $k = 0, 1, 2, \ldots$

So n even: $I = \frac{1}{2k}\sin 2kx\Big|_{\pi/2}^{\pi}$

$= \frac{1}{2k}\left(\sin 2\pi k - \sin \pi k\right) = 0$ for all k.

for n even : $I = \frac{1}{2k+1} \sin(2k+1)x \Big|_{\pi/2}^{\pi}$

$= \frac{1}{2k+1}\left(\sin(2k+1)\pi - \sin(2k+1)\frac{\pi}{2}\right)$

The 1st term in the Brackets is always 0 for all k.
The 2nd term in The Brackets is alternates between ± 1 depending on whether k is even or odd

So value is 1 when $k = 0, 2, 4, \ldots$
& -1 when $k = 1, 3, 5, \ldots$

So The Three values are $I = -\frac{1}{n}, 0, \frac{1}{n}$

(c) let $I = \int_0^{\pi/4} \frac{2\cos x - \sin x}{2\sin x + \cos x}\, dx$

let $u = 2\sin x + \cos x$, $\therefore du = 2\cos x - \sin x \, dx$

And if $x = 0$, $u = 1$
$x = \frac{\pi}{4}$, $u = \frac{3\sqrt{2}}{2}$

So $I = \int_1^{3\sqrt{2}/2} \frac{1}{u}\, du = \left[\ln|u|\right]_1^{3\sqrt{2}/2}$

$= \ln \frac{3\sqrt{2}}{2} - \ln 1$

$\simeq 0.752$

(19) $$\frac{x-2}{2x^2-x-3} = \frac{x-2}{(2x-3)(x+1)} = \frac{A}{2x-3} + \frac{B}{x+1}$$

So $x-2 = A(x+1) + B(2x-3)$

When $x=-1$: $-3 = -5B \Rightarrow B = 3/5$

$x = \frac{3}{2}$: $-\frac{1}{2} = \frac{5}{2}A \Rightarrow A = -\frac{1}{5}$

So $$\frac{x-2}{2x^2-x-3} = \frac{-1/5}{2x-3} + \frac{3/5}{x+1}$$

$$\therefore \quad I = \int_2^3 \frac{x-2}{2x^2-x-3}\,dx = \int_2^3 \frac{-1/5}{2x-3} + \frac{3/5}{x+1}\,dx$$

$$= \left[-\frac{1}{5}\cdot\frac{1}{2}\ln|2x-3| + \frac{3}{5}\ln|x+1|\right]_2^3$$

$$= \left[\frac{1}{5}\ln\left|\frac{(x+1)^3}{(2x-3)^{1/2}}\right|\right]_2^3$$

$$= \frac{1}{5}\left(\ln\frac{4^3}{\sqrt{3}} - \ln\frac{3^3}{1}\right)$$

$$= \frac{1}{5}\ln\left(\left(\frac{4}{3}\right)^3\frac{1}{\sqrt{3}}\right) \simeq 0.062748$$

which is the same as the answer in the book.